The Category of Categories is Cartesian Closed

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The category of categories is cartesian closed. This means amongst others that $curry(\dagger)$ is well defined for any bi-functor $\dagger$, having the properties that we expect it to have. The *sectioning* notation $x\dagger$ may be used to denote "x subject to $curry(\dagger)$" (for object or morphism x); it follows that $f\dagger$ is a natural transformation from $A\dagger$ to $B\dagger$, whenever $f : A \to B$.

Introduction

The category of categories, $\mathcal{C}$, has all (small?) categories as objects and all functors as morphisms. Cartesian closedness of $\mathcal{C}$ means

- there exists a category $\mathbf{1}$ that is *final* in $\mathcal{C}$,
- for any two categories $\mathbf{A}$ and $\mathbf{B}$ there exists a category $\mathbf{A} \times \mathbf{B}$ and suitable projection functors that together constitute a *product* in $\mathcal{C}$, and
- for any two categories $\mathbf{A}$ and $\mathbf{B}$ there exists a category $\mathbf{A}{\rightarrow}\mathbf{B}$ and functor $@_{\mathbf{A},\mathbf{B}}$: $\mathbf{A} \times (\mathbf{A}{\rightarrow}\mathbf{B}) \to \mathbf{B}$ and, for any functor $\dagger : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$, a functor $\dagger\hat{} : \mathbf{A} \to (\mathbf{B}{\rightarrow}\mathbf{C})$ that together constitute an *exponent* in $\mathcal{C}$.

Once the formal requirements are laid down, most of the definitions are straightforward and present no surprises if the point-wise construction of the final object, the products, and the exponents within **Set** are known. Also, the verification that the required typing and equations are fullfilled is a matter of routine. The "only" difference with **Set** is this: in **Set** a morphism is just a single function, whereas in $\mathcal{C}$ a morphism is a functor and therefore *both* a function from objects to objects *and* a function from morphisms to morphisms.

Let us consider the construction for exponents in some more detail. We shall use the following notation and naming convention, unless stated explicitly otherwise.
$\mathbf{A}, \mathbf{B}, \ldots$ vary over categories (i.e. $\mathbf{A} \in \mathrm{Obj}(\mathcal{C})$);
$\mathsf{F}, \mathsf{G}, \ldots$ vary over functors, typically $\mathsf{F} : \mathbf{A} \to \mathbf{B}$ (i.e., $\mathsf{F} \in Hom_{\mathcal{C}}(\mathbf{A}, \mathbf{B})$);
$x.\mathsf{F}$ denotes "x subject to F", and $x.(\mathsf{F}; \mathsf{G}) = (x.\mathsf{F}).\mathsf{G}$;

$\dagger : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ in $\mathcal{C}$ and $x \dagger y = (x, y).\dagger$;
$A, A', \ldots$ vary over objects of $\mathbf{A}$ (i.e., $A \in \mathrm{Obj}(\mathbf{A})$) and so on;
$f, g, \ldots$ vary over morphisms, typically $f : A \to A'$ in $\mathbf{A}$ (i.e., $f \in Hom_{\mathbf{A}}(A, A')$) and $g : B \to B'$ in $\mathbf{B}$;
composition of morphisms in $\mathbf{A}$ and so on is denoted $f; f'$.

Exponents

Exponent, currying Given categories $\mathbf{A}$ and $\mathbf{B}$ we define the category $\mathbf{A}{\to}\mathbf{B}$ to be the well-known category of functors from $\mathbf{A}$ to $\mathbf{B}$ whose morphisms are natural transformations. Given a (bi-) functor $\dagger : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ we define the (mono-)functor $\dagger\hat{} : \mathbf{A} \to (\mathbf{B}{\to}\mathbf{C})$ as follows.

$$
\begin{array}{lrcl}
 & A.\dagger\hat{} & = & \text{the functor from } \mathbf{B} \text{ to } \mathbf{C} \text{ given by} \\
 & & & B.(A.\dagger\hat{}) \quad = \quad A \dagger B \\
(1) & & & g.(A.\dagger\hat{}) \quad = \quad id_A \dagger g \quad : \quad B.(A.\dagger\hat{}) \to B'.(A.\dagger\hat{}) \\
 & f.\dagger\hat{} & = & \text{the natural transformation from } A.\dagger\hat{} \text{ to } A'.\dagger\hat{} \text{ given by} \\
 & & & B.(f.\dagger\hat{}) \quad = \quad f \dagger id_B \quad : \quad A \dagger B \to A' \dagger B \text{ in } \mathbf{C} \,;
\end{array}
$$

The requirements for $A.\dagger\hat{}$ to be a functor, and for $f.\dagger\hat{}$ to be a natural transformation, are easily verified. We can extend the above definition of $f.\dagger\hat{}$ (as a mapping from objects to morphisms) with a mapping from morphisms to morphisms as follows. (Here $g \bullet \varphi$ denotes "g subject to φ".)

$$
\begin{array}{lrcl}
 & g \bullet (f.\dagger\hat{}) & = & f \dagger g \\
 & & = & (id_A \dagger g); (f \dagger id_{B'}) \\
(2) & & = & g.(A\dagger\hat{}); B'.(f\dagger\hat{}) \\
 & & = & B.(f\dagger\hat{}); g.(A'\dagger\hat{}) \,;
\end{array}
$$

This is no surprise since we can do so in general for any natural transformation $\varphi : \mathsf{F} \to \mathsf{G}$ in $\mathbf{B}{\to}\mathbf{C}$ (with $\mathsf{F}, \mathsf{G} : \mathbf{B} \to \mathbf{C}$ in $\mathcal{C}$):

$$(3) \qquad g.\mathsf{F}; \varphi_{B'} \quad = \quad \varphi_B; g.\mathsf{G} \quad =: \quad g \bullet \varphi$$

for any $g : B \to B'$ in $\mathbf{B}$.

Sectioning We may use the notation $x\dagger$ for $x.\dagger\hat{}$. It has been defined above for both objects x and morphisms x , and we have seen that $A\dagger$ is a functor and $f\dagger$ is a natural transformation.

When object A in $\mathbf{A}$ is also used to denote the identity morphism $id_A : A \to A$ and the constant functor $A^\bullet : \mathbf{X} \to \mathbf{A}$ (mapping an object to A and a morphism to id_A), then we can summarize all four definitions of $\dagger\hat{}$ by

$$(4) \qquad y.\,(x\dagger) \quad = \quad x \dagger y \quad \text{in } \mathbf{C}$$

for any object and morphism x in $\mathbf{A}$ and any object and morphism y in $\mathbf{B}$. (Notice that there is a syntactic ambiguity in $f; A.\mathsf{F}$ and $(A.\mathsf{F}); \mathsf{F}'$ but no semantic ambiguity, since $id_A.\mathsf{F} = id_{A.\mathsf{F}}$.)

Evaluation We also need to define for any two objects $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}$ an evaluation functor $@_{\mathbf{A},\mathbf{B}} : \mathbf{A} \times (\mathbf{A}{\to}\mathbf{B}) \to \mathbf{B}$. As a mapping on objects its definition suggests itself; as a mapping on morphisms it might be a very little bit surprising.

$$
(5) \qquad \begin{array}{lcl} (A, \mathsf{F}).@ & = & A.\mathsf{F} \quad \text{in } \mathbf{B} \\ (f, \varphi).@ & = & f \bullet \varphi \quad : \quad A.\mathsf{F} \to A'.\mathsf{F} \qquad (= f.\mathsf{F}; \varphi = \varphi; f.\mathsf{G}) \ ; \end{array}
$$

for $f : A \to A'$ in $\mathbf{A}$ and $\varphi : \mathsf{F} \to \mathsf{G}$ in $\mathbf{A}{\to}\mathbf{B}$. In order to fully complete the proof that these constructions do constitute an exponent, the following equivalence has to be satisfied:

$$
(6) \qquad \mathsf{F} = \dagger\hat{} \;\; \equiv \;\; \mathsf{F} \times \mathrm{I}_{\mathbf{B}\to\mathbf{C}}; @_{\mathbf{B},\mathbf{C}} = \dagger
$$

for all $\mathsf{F} : \mathbf{A} \to \mathbf{B}$ in $\mathcal{C}$. Since @ is defined pointwise one can easily check the equivalence by extensionality.

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