Map-functor Factorized

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It is well known that any initial data type comes equipped with a so-called map-functor. We show that any such map-functor is the composition of two functors, one of which is —closely related to— the data type functor, and the other is —closely related to— the function μ (that for any functor F yields an initial F-algebra, if it exists).

Notation

Let K be a category, and $\mathsf{F} : K \to K$ be an endo-functor on K. Then $\mu\mathsf{F}$ denotes "the" initial F-algebra over K, if it exists. Further, $\mathcal{F}(K)$ is the category of endo-functors on K whose morphisms are, as usual, natural transformations; and $\mathcal{F}_\mu(K)$ denotes the full sub-category of $\mathcal{F}(K)$ whose objects are those functors F for which $\mu\mathsf{F}$ exists.

For mono-functors F, G and bi-functor $\dagger$ we define the composition FG by $x(\mathsf{FG}) = (x\mathsf{F})\mathsf{G}$, and we denote by $\mathsf{F} \dagger \mathsf{G}$ the mono-functor defined by $x(\mathsf{F} \dagger \mathsf{G}) = x\mathsf{F} \dagger x\mathsf{G}$. Object A when used as a functor is defined by $xA = A$ for any object x and $fA = id_A$ for any morphism f. (An alternative notation for $A \dagger \mathrm{I}$ is the 'section' $A\dagger$.) In the examples we assume that $\times, \grave{\pi}, \acute{\pi}, \vartriangle$ form a product, and $+, \grave{\imath}, \acute{\imath}, \triangledown$ a co-product.

Making μ into a functor

We define a functor $\_^\mu : \mathcal{F}_\mu(K) \to K$ that is closely related to μ, and has therefore a closely related notation. For any $\mathsf{F}, \mathsf{G} \in Obj(\mathcal{F}_\mu(K))$ and $\varphi : \mathsf{F} \dot{\to} \mathsf{G}$ we put

(1) $\quad \mathsf{F}^\mu \quad = \quad \text{target of } \mu\mathsf{F}$

(2) $\quad \varphi^\mu \quad = \quad (\!|\varphi; \mu\mathsf{G}|\!)_\mathsf{F} \quad : \quad \mathsf{F}^\mu \to \mathsf{G}^\mu$.

Notice that by (1) we have $\mu\mathsf{F} : \mathsf{F}^\mu\mathsf{F} \to \mathsf{F}^\mu$. (Some authors in the Squiggol community are used to define $(L, in) = (\mathsf{F}^\mu, \mu\mathsf{F})$.) The instance of φ that has to be taken in the right-hand side of (2) is $\varphi_{\mathsf{G}^\mu} : \mathsf{G}^\mu\mathsf{F} \to \mathsf{G}^\mu\mathsf{G}$; the typing $\varphi^\mu : \mathsf{F}^\mu \to \mathsf{G}^\mu$ is then easily verified. In order to prove that $\_^\mu$ satisfies the two other functor axioms, we present a lemma first.

(3) Lemma *For* $\varphi : \mathsf{F} \dot{\to} \mathsf{G}$ *and* $\psi : A\mathsf{G} \to A$,

(4) $\quad (\!|\varphi; \psi|\!)_\mathsf{F} \quad = \quad (\!|\varphi; \mu\mathsf{G}|\!); (\!|\psi|\!)_\mathsf{G}$.

Proof (Within this proof we use the law names and notation of Fokkinga & Meijer [1]. The reader may easily verify the steps by unfolding $f : \varphi \stackrel{\mathsf{F}}{\to} \psi$ into $\varphi ; f = f\mathsf{F} ; \psi$, and using $\varphi : \mathsf{F} \dot{\to} \mathsf{G} \;\equiv\; (\forall f :: f\mathsf{F} ; \varphi = \varphi ; f\mathsf{G})$.)

$$
\begin{array}{ll}
 & \text{required equality} \\
\Leftarrow & \qquad \textsc{Fusion} \\
 & (\!|\psi|\!)_{\mathsf{G}} : \varphi ; \mu\mathsf{G} \stackrel{\mathsf{F}}{\to} \varphi ; \psi \\
\Leftarrow & \qquad \textsc{Ntrf to Homo},\ \varphi : \mathsf{F} \dot{\to} \mathsf{G} \\
 & (\!|\psi|\!)_{\mathsf{G}} : \mu\mathsf{G} \stackrel{\mathsf{G}}{\to} \psi \\
\equiv & \qquad \textsc{Cata Homo} \\
 & \mathsf{true}.
\end{array}
$$

(End of proof)

It is now immediate that $\_^{\mu}$ distributes over composition. For $\varphi : \mathsf{F} \dot{\to} \mathsf{G}$ and $\psi : \mathsf{G} \dot{\to} \mathsf{H}$ we have $\varphi ; \psi : \mathsf{F} \dot{\to} \mathsf{H}$ and

$$
\begin{array}{ll}
 & (\varphi ; \psi)^{\mu} \\
= & \\
 & (\!|\varphi ; \psi ; \mu\mathsf{H}|\!)_{\mathsf{F}} \\
= & \qquad \text{Lemma (3), noting that } \psi ; \mu\mathsf{H} : \mathsf{H}^{\mu}\mathsf{G} \to \mathsf{H}^{\mu} \\
 & (\!|\varphi ; \mu\mathsf{G}|\!)_{\mathsf{F}} ; (\!|\psi ; \mu\mathsf{H}|\!)_{\mathsf{G}} \\
= & \\
 & \varphi^{\mu} ; \psi^{\mu} .
\end{array}
$$

It is also clear that $id^{\mu} = id$. Thus, $\_^{\mu}$ is a functor, $\_^{\mu} : \mathcal{F}_{\mu}(K) \to K$.

(5) Remark Another corollary of the lemma is this: for $\varphi : \mathsf{F} \dot{\to} \mathsf{G}$ we have that $\varphi^{\mu} ; f$ is a catamorphism whenever f is a catamorphism. (The typing determines that the former is an F-catamorphism, and the latter a G-catamorphism.) □

Let us look at some $\varphi : \mathsf{F} \dot{\to} \mathsf{G}$ and see what φ^{μ} is.

Example Probably the most simple, non-trivial, choice is $\mathsf{F}, \mathsf{G} := \mathbf{1} + A \times \mathrm{I},\ \mathbf{1} + \mathrm{I}$ and $\varphi := id + \acute{\pi}$. Notice that F^{μ} = the (set L of) cons-lists and $\mu\mathsf{F} = nil \triangledown cons$, G^{μ} = the (set $\mathbb{N}$ of) naturals and $\mu\mathsf{G} = zero \triangledown suc$. We find

$$(6) \qquad \varphi^{\mu} \;=\; (\!|id + \acute{\pi} ; zero \triangledown suc|\!)_{\mathsf{F}} \;=\; size \;:\; L \to \mathbb{N} .$$

□

Example Another non-trivial choice is $\mathsf{F} = \mathsf{G} = A + \mathrm{I\!I}$, so that $\mathsf{F}^{\mu} = \mathsf{G}^{\mu}$ = the (set of) non-empty binary join trees over A , and $\mu\mathsf{F} = tip \triangledown join$. Apart from the trivial $id : \mathsf{F} \dot{\to} \mathsf{G}$, we have $\varphi := id + swap : \mathsf{F} \dot{\to} \mathsf{G}$ where $swap = \acute{\pi} \vartriangle \grave{\pi}$. We have

$$(7) \qquad \varphi^{\mu} \;=\; (\!|id + swap ; tip \triangledown join|\!) \;=\; swap/ \;=\; reverse .$$

Since $\_^\mu$ is a functor, we have a simple proof that *reverse* is its own inverse:

$$\begin{array}{ll} & reverse ; reverse \\ = & \\ & \varphi^\mu ; \varphi^\mu \\ = & \quad \text{functor axiom} \\ & (\varphi ; \varphi)^\mu \\ = & \quad \text{easy: } swap ; swap = id \\ & id^\mu \\ = & \\ & id . \end{array}$$

Notice also that by Remark (5), $reverse ; f$ is a catamorphism whenever f is. □

Example Let $\dagger$ be a bi-functor and let $\mathsf{F} = A \dagger \mathrm{I}$ and $\mathsf{G} = \mathbf{1} \dagger \mathrm{I}$. Take $\varphi = ! \dagger id : A \dagger \mathrm{I} \dot{\to} \mathbf{1} \dagger \mathrm{I}$. Then

(8) $\varphi^\mu \quad = \quad (\!| ! \dagger id ; \mu(\mathbf{1} \dagger \mathrm{I}) |\!)_{A \dagger \mathrm{I}} \quad = \quad shape \quad (= \quad !\text{-map})\,.$

□

Factorizing map-functors

Let $\dagger$ be any bi-functor for which $\mu(A \dagger \mathrm{I})$ exists for all A. Recall that *the map-functor induced by* $\dagger$, $\_^\varpi$ say, is defined by

$$\begin{array}{lcl} A^\varpi & = & \text{target of } \mu(A \dagger \mathrm{I}) \\ f^\varpi & = & (\!| f \dagger id ; \mu(B \dagger \mathrm{I}) |\!)_{A \dagger \mathrm{I}} \quad : \quad A^\varpi \to B^\varpi \end{array}$$

for $f : A \to B$. We shall now define a functor $\_^\dagger : K \to \mathcal{F}_\mu(K)$ in such a way that composed with $\_^\mu : \mathcal{F}_\mu(K) \to K$ it equals the map-functor $\_^\varpi : K \to K$. To this end define

$$\begin{array}{lcl} A^\dagger & = & A \dagger \mathrm{I} \\ f^\dagger & = & f \dagger id \quad : \quad A \dagger \mathrm{I} \dot{\to} B \dagger \mathrm{I} \qquad (\text{with } (f \dagger id)_C = f \dagger id_C) \end{array}$$

for any $f : A \to B$. (That $f^\dagger$ is a natural transformation is easily verified; it also follows from laws Ntrf Triv, Ntrf Id, Ntrf Bi-Distr from Fokkinga & Meijer [1].) Indeed

$$\begin{array}{lcl} A^{\dagger\mu} & = & (A \dagger \mathrm{I})^\mu \quad = \quad A^\varpi \\ f^{\dagger\mu} & = & (f \dagger id)^\mu \quad = \quad (\!| f \dagger id ; \mu(B \dagger \mathrm{I}) |\!)_{A \dagger \mathrm{I}} \quad = \quad f^\varpi . \end{array}$$

So $\varpi = \dagger\,\mu$.

Remark It can be shown that $\_^\dagger$ is just $curry(\dagger)$. (Here $curry(\_)$ is the well-defined functor from the category $\mathbf{A} \times \mathbf{B} \to \mathbf{C}$ to the category $\mathbf{A} \to (\mathbf{B} \to \mathbf{C})$, where each arrow denotes a category of functors with natural transformations as morphisms.) Thus, given bi-functor $\dagger$, we can express its map-functor without further auxiliary definitions as $curry(\dagger)$ composed with $^\mu$. □

References

[1] M.M. Fokkinga and E. Meijer. Program calculation properties of continuous algebras. December 1990. CWI, Amsterdam.