Restricted Choices

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Problem statement. Given are three numbers N, p, q with $0 \le p \le 3N$ and $0 \le q < N$. The problem is to determine the number of ways that one can choose p elements from a sequence of three sets of size N in such a way that in the first set more than q elements are chosen and in each of the second and third set at most q. So, identifying a set of size N with numbers $1 \mathinner{..} N$, the problem is to determine the following number:

$$(1) \quad \#\{A, B, C : 1 \mathinner{..} N \mid \#A + \#B + \#C = p \;\;\wedge\;\; q < \#A \;\;\wedge\;\; \#B \le q \;\;\wedge\;\; \#C \le q\}$$

Problem solving strategy. The idea is to massage the set in the problem specification (1) according to the following law (assuming P_i and D_j $(j \ge i)$ have no free variables in common):

$$(2) \quad \{D_1; D_2; D_3 \mid P_1 \wedge P_2 \wedge P_3 \bullet E\} \;=\; \bigcup\{D_1 \mid P_1 \bullet \bigcup\{D_2 \mid P_2 \bullet \bigcup\{D_3 \mid P_3 \bullet \{E\}\}\}\}$$

This can be followed by pushing the cardinality $\#$ through the big-union $\bigcup$ as follows (assuming that E denotes a set and $\forall\, D; D' \mid P \wedge P' \wedge D \ne D' \bullet E \cap E' = \varnothing$):

$$(3) \quad \#\bigcup\{D \mid P \bullet E\} \;=\; \sum D \mid P \bullet \#E$$

And finally, if in the previous line the D, P, E have a suitable form, then a simplification of the right-hand side is possible using the binomial $\binom{n}{k}$ (which equals $\#\{A : 1 \mathinner{..} n \mid \#A = k\}$):

$$(4) \quad \sum A : 1 \mathinner{..} N \mid low \le \#A \le up \bullet f(\#A) \;=\; \sum a : low \mathinner{..} up \bullet \binom{N}{a} \times f(a)$$

Fully spelled out, and leaving the domain $1 \mathinner{..} N$ implicit and writing a, b, c for $\#A, \#B, \#C$, this strategy reads as follows:

$$\#\{A, B, C \mid \text{“condition on } a, b, c\text{”} \bullet (A, B, C)\}$$

$=$ defining suitable lowerbounds and upperbounds

$$\#\{A, B, C \mid \; L \le a \le U \;\wedge\; L_a \le b \le U_a \;\wedge\; L_{a,b} \le c \le U_{a,b} \;\bullet\; (A, B, C)\}$$

$=$ law (2)

$$\#\bigcup\{A \mid L \le a \le U \bullet \bigcup\{B \mid L_a \le b \le U_a \bullet \bigcup\{C \mid L_{a,b} \le c \le U_{a,b} \bullet \{(A, B, C)\}\}\}\}$$

$=$ repeatedly law (3)

$$\sum A \mid L \le a \le U \bullet \sum B \mid L_a \le b \le U_a \bullet \sum C \mid L_{a,b} \le c \le U_{a,b} \bullet \#\{(A, B, C)\}$$

$=$ three times law (4), and “singleton has size one”

$$\sum a : L \mathinner{..} U \bullet \left(\binom{N}{a} \times \sum b : L_a \mathinner{..} U_a \bullet \left(\binom{N}{b} \times \sum c : L_{a,b} \mathinner{..} U_{a,b} \bullet \binom{N}{c} \times 1\right)\right)$$

$=$ notational change; scope of $\sum$ extends to the right as far as possible

$$\textstyle\sum_{a:L..U} \;\; \binom{N}{a} \times \sum_{b:L_a..U_a} \binom{N}{b} \times \sum_{c:L_{a,b}..U_{a,b}} \binom{N}{c}$$

Derivation of the bounds. In view of aimed applications of law (4), we try to massage the constraints on $\#A, \#B, \#C$ in the problem specification (1) into lowerbounds and upperbounds for these sizes. For readability we put $a = \#A$, $b = \#B$, and $c = \#C$. First we derive bounds for c, then for b (without using c), and finally for a (without using c, b):

“given constraint on a, b, c in (1)”

$\equiv$

$q < a \wedge$

$b \le q \wedge$

$c \le q \wedge$

$a + b + c = p$

$\equiv$ global knowledge: size A at most N, and sizes B and C at least 0

$q < a \le N \wedge$

$0 \le b \le q \wedge$

$0 \le c \le q \wedge$

$a + b + c = p$

$\equiv$ last line defines c, elimination of c in remaining clauses

$q < a \le N \wedge$

$0 \le b \le q \wedge$

$0 \le p{-}a{-}b \le q \wedge$

$c = p{-}a{-}b$

$\equiv$ reformulate one-but-last line as bounds for b

$q < a \le N \wedge$

$0 \le b \le q \wedge$

$p{-}a{-}q \le b \le p{-}a \wedge$

$c = p{-}a{-}b$

$\equiv$ taking middle lines together

$q < a \le N \wedge$

$\max(0, p{-}a{-}q) \le b \le \min(q, p{-}a) \wedge$

$c = p{-}a{-}b$

$\equiv$ taking apart the lower and upperbound of b

$q < a \le N \wedge$

$\max(0, p{-}a{-}q) \le \min(q, p{-}a) \wedge$

$\max(0, p{-}a{-}q) \le b \le \min(q, p{-}a) \wedge$

$c = p{-}a{-}b$

$\equiv$ elaboration second line into four comparisons

$q < a \le N \wedge$

$0 \le q \;\wedge\; p{-}a{-}q \le q \;\wedge\; 0 \le p{-}a \;\wedge\; p{-}a{-}q \le p{-}a \wedge$

$\max(0, p{-}a{-}q) \le b \le \min(q, p{-}a) \wedge$

$c = p{-}a{-}b$

$\equiv$ simplification second line using given bounds of p and q

$q < a \le N \land$

$p{-}2q \le a \;\land\; a \le p \land$

$\max(0, p{-}a{-}q) \le b \le \min(q, p{-}a) \land$

$c = p{-}a{-}b$

$\equiv$ taking first two lines together

$\max(q{+}1, p{-}2q) \le a \le \min(N, p) \land$

$\max(0, p{-}a{-}q) \le b \le \min(q, p{-}a) \land$

$c = p{-}a{-}b$

$\equiv$ taking apart the lower and upperbound of a

$\max(q{+}1, p{-}2q) \le \min(N, p) \land$

$\max(q, p{-}2q) \le a \le \min(N, p) \land$

$\max(0, p{-}a{-}q) \le b \le \min(q, p{-}a) \land$

$c = p{-}a{-}b$

$\equiv$ simplification first line (via four comparisons), using the bounds for p, q

$q < p \le N{+}2q \land$

$\max(q{+}1, p{-}2q) \le a \le \min(N, p) \land$

$\max(0, p{-}a{-}q) \le b \le \min(q, p{-}a) \land$

$c = p{-}a{-}b$

(It follows that $0 < p$.) For readability we give names to the bounds for a, b, c:

$$L = \max(q{+}1, p{-}2q) \qquad U = \min(N, p)$$
$$L_a = \max(0, p{-}a{-}q) \qquad U_a = \min(q, p{-}a)$$
$$L_{a,b} = p{-}a{-}b = U_{a,b}$$

The solution. Applying the strategy, and using the bounds just derived, we get:

$\#\{A, B, C : 1..N \mid \#A + \#B + \#C = p \;\land\; q < \#A \;\land\; \#B \le q \;\land\; \#C \le q\}$

$=$ notational definition

$\#\{A, B, C : 1..N \mid$ "same constraint as in previous line" $\bullet\ (A, B, C)\}$

$=$ following the strategy outlined above

$\sum_{a:\, L..U \;\land\; q<p\le N+2q} \binom{N}{a} \times \sum_{b:\; L_a..U_a} \binom{N}{b} \times \sum_{c:\; L_{a,b}..U_{a,b}} \binom{N}{c}$

$=$ bounds for c coincide

$\sum_{a:\, L..U \;\land\; q<p\le N+2q} \binom{N}{a} \times \sum_{b:\; L_a..U_a} \binom{N}{b} \times \binom{N}{L_{a,b}}$

$=$ taking apart the condition on a that does not depend on a

0 if not $(q < p \le N{+}2q)$ else $\sum_{a:\; L..U} \binom{N}{a} \times \sum_{b:\; L_a..U_a} \binom{N}{b} \times \binom{N}{L_{a,b}}$

$=$ substituting definitions for L, U, L_a, U_a, notational change

0 if not $(q < p \le N{+}2q)$ else

$$\sum_{a=\max(q+1,p-2q)}^{\min(N,p)} \binom{N}{a} \times \sum_{b=\max(0,p-a-q)}^{\min(q,p-a)} \binom{N}{b} \times \binom{N}{p-a-b}$$