Dyads, a generalisation of monads

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The concept of *dyad* is defined as the least common generalisation of *monads* and *co-monads*. So, taking some of the ingredients to be the identity, the concept specialises to the concept of monad, and taking other ingredients to be the identity it specialises to co-monads. Except for one axiom, all have a nice ("natural") form.

Introduction: monads

Let us first describe one way in which monads can be motivated. We shall motivate dyads by a similar argument later. (Other descriptions have been given by Barr and Wells [1] and Wadler [2].) We use a fairly standard notation: $\mathcal{A}, \mathcal{B}, \ldots$ denote categories, $a, b, \ldots$ objects, $f, g, \ldots$ arrows, $F, G, \ldots$ functors, greek letters denote natural transformations, and $x, y, \ldots$ various things. Composition is denoted in diagrammatic order: $f \mathbin{;} g$ is normally written $g \circ f$.

Example. Let L be the list functor: La is the set of lists over a, and for $f\colon a \to b$ we have $Lf\colon La \to Lb$ as the well-known f-map. Given list producing functions $f\colon a \to Lb$ and $g\colon b \to Lc$, we often see the list producing "composition":

$$f \mathbin{;} Lg \mathbin{;} +\!\!+/ \quad : \quad a \to Lc \quad .$$

Here $+\!\!+/\colon LL \mathrel{\dot\to} L$ is the flattening, or concatenation, of lists of list into lists. Functions of type $a \to Lb$, for varying a, b, turn up frequently in actual programming. For example, unconditional list comprehensions can be described in that way:

$$x \mapsto [z|\ y \leftarrow f\,x;\ z \leftarrow g\,y] \quad = \quad f \mathbin{;} Lg \mathbin{;} +\!\!+/ \quad .$$

Notice, moreover, the existence of a particularly nice list producing function of type $a \to Lb$ with $a = b$, namely the singleton former $\eta = \lambda x :: [x]$. It has the property that $f \mathbin{;} L\eta \mathbin{;} +\!\!+/ = f = \eta \mathbin{;} Lf \mathbin{;} +\!\!+/$.

Generalisation. The situation above can be described more elegantly in categorical terms as follows. Let F be an endofunctor. Under what conditions can we consider arrows of type $f\colon a \to Fb$, for varying a, b, to be arrows of type $a \to b$ in another category? This question means, amongst others, that there must be a way to "compose" arrows $f\colon a \to Fb$ and $g\colon b \to Fc$ into an arrow of type $a \to Fc$, and that this composition is associative, and that there exists a function $\eta_a\colon a \to Fa$ for each a that is the identity for the new "composition".

The new category is known as the Kleisli category, and we shall indicate it by $\mathcal{K}(F)$, or simply $\mathcal{K}$ if the functor F is understood. The construction of $\mathcal{K}$ is straightforward. Define:

$$
\begin{array}{lcl}
a \text{ object in } \mathcal{K} & \equiv & a \text{ object in the given category} \\
f \text{ arrow in } \mathcal{K} & \equiv & f \text{ arrow in the given category} \\
 & & \quad \text{of type } a \to Fb \text{ for some } a, b \\
f\colon a \to_{\mathcal{K}} b & \equiv & f\colon a \to Fa \\
f \,;_{\mathcal{K}}\, g & = & f \,;\, Fg \,;\, \mu \qquad \text{whenever } f\colon a \to Fb, \quad g\colon b \to Fc \\
id_{\mathcal{K},a} & = & \eta_a
\end{array}
$$

where natural transformations μ and η are assumed to exist:

$$
\begin{array}{lcl}
\mu & : & FF \dot{\to} F \\
\eta & : & I \dot{\to} F \quad .
\end{array}
$$

The associativity of composition $;_{\mathcal{K}}$, and the neutrality of $id_{\mathcal{K}}$ for this composition, are equivalent to the following three properties of μ and η:

$$
\begin{array}{lclcl}
F\mu \,;\, \mu & = & \mu F \,;\, \mu & & \\
F\eta \,;\, \mu & = & \eta F \,;\, \mu & = & idF \quad .
\end{array}
$$

The first two of these say that $F\mu$ equals μF, and $F\eta$ equals ηF, when followed by μ. Indeed, a proof of associativity reads:

$$
\begin{array}{ll}
 & (f \,;_{\mathcal{K}}\, g) \,;_{\mathcal{K}}\, h = f \,;_{\mathcal{K}}\, (g \,;_{\mathcal{K}}\, h) \\
\equiv & \qquad \text{definition } ;_{\mathcal{K}} \\
 & (f \,;\, Fg \,;\, \mu) \,;\, Fh \,;\, \mu = f \,;\, F(g \,;\, Fh \,;\, \mu) \,;\, \mu \\
\equiv & \qquad \text{category, functor} \\
 & f \,;\, Fg \,;\, \mu \,;\, Fh \,;\, \mu = f \,;\, Fg \,;\, FFh \,;\, F\mu \,;\, \mu \\
\equiv & \qquad \text{in lhs: naturality } \mu \\
 & f \,;\, Fg \,;\, FFh \,;\, \mu F \,;\, \mu = f \,;\, Fg \,;\, FFh \,;\, F\mu \,;\, \mu \\
\equiv & \qquad \text{for } \Leftarrow\text{: Leibniz; for } \Rightarrow\text{: instantiate with } f, g, h \text{ all equal to } id \\
 & \mu F \,;\, \mu = F\mu \,;\, \mu \quad .
\end{array}
$$

A proof of neutrality of $id_{\mathcal{K}}$ for $;_{\mathcal{K}}$ reads:

$$
\begin{array}{ll}
 & f \;;_{\mathcal{K}}\; id_{\mathcal{K}} = f = id_{\mathcal{K}} \;;\; f \\
\equiv & \qquad \text{definition } ;_{\mathcal{K}} \text{ and } id_{\mathcal{K}} \\
 & f \;;\; F\eta \;;\; \mu = f = \eta \;;\; Ff \;;\; \mu \\
\equiv & \qquad \text{naturality } \eta \\
 & f \;;\; F\eta \;;\; \mu = f = f \;;\; \eta F \;;\; \mu \\
\equiv & \qquad \text{for } \Leftarrow\text{: Leibniz; for } \Rightarrow\text{: instantiate with } f = id \\
 & F\eta \;;\; \mu = idF = \eta F \;;\; \mu \quad .
\end{array}
$$

The three remaining category axioms for $\mathcal{K}$ read:

$$
\begin{array}{lll}
f\colon\ a \to_{\mathcal{K}} b, \quad g\colon\ b \to_{\mathcal{K}} c & \Rightarrow & f \;;_{\mathcal{K}}\; g\colon\ a \to_{\mathcal{K}} c \\
id_{\mathcal{K}}\colon\ a \to_{\mathcal{K}} a & & \\
f\colon\ a \to_{\mathcal{K}} b, \quad f\colon\ a' \to_{\mathcal{K}} b' & \Rightarrow & a = a' \quad \wedge \quad b = b' \quad .
\end{array}
$$

The first two of these are evidently true; the latter one isn't true in general. So the construction yields a pre-category, and by a standard construction one obtains a category, the Kleisli category $\mathcal{K}$.

In summary, the ingredients that enable the construction of a Kleisli category are:

$$
\begin{array}{l}
(F, \quad \mu\colon\ FF \dot{\to} F, \quad \eta\colon\ Id \dot{\to} F) \\
\text{satisfying} \\
F\mu \;;\; \mu \quad = \quad \mu F \;;\; \mu \\
F\eta \;;\; \mu \quad = \quad \eta F \;;\; \mu \quad = \quad idF \quad .
\end{array}
$$

Such a triple is known as a **monad**.

A generalisation wanted

Loosely formulated the above construction of the Kleisli category enables us to "compose" arrows of type $a \to Fb$ (considering them to be of type $a \to' b$) for fixed F and varying a, b. Doaitse Swierstra posed the problem of "composing" arrows of type $a \times b \to Fc$, for fixed F and varying a, b, c. The Kleisli construction doesn't help here.

Abstracting a little bit, the new problem is to construct a Kleisli-like category for arrows of type $Fa \to Gb$ (considering them to be of type $a \to' b$), for fixed F, G and varying a, b. Taking $F = Id$ we get the original problem that is solved by Kleisli's construction and the existence of a monad for G. Taking $G = Id$ we get the dual problem; solved by the dual of Kleisli's construction and the existence of a co-monad for F. Thus our current *problem* formulation is the least common generalisation of the "Kleisli problem" and its dual; and our *solution* will be the least common generalisation of the Kleisli construction and its dual. We will term the ingredients of the solution a **dyad**: the least common generalisation of a monad and a co-monad.

Dyads

Let F, G be functors with a common source and a common target. We will construct a category $\mathcal{D}(F,G)$, or simply $\mathcal{D}$, whose arrows (of type $a \to_{\mathcal{D}} b$) are the given arrows of type $Fa \to Gb$. To this end we define:

$$\begin{array}{lcll}
a \text{ object in } \mathcal{D} & \equiv & a \text{ object in source category of } F, G & \\
f \text{ arrow in } \mathcal{D} & \equiv & f \text{ arrow in target category of } F, G & \\
 & & \quad \text{of type } Fa \to Gb \text{ for some } a, b & \\
f\colon a \to_{\mathcal{D}} b & \equiv & f\colon Fa \to Gb & \\
f \;;_{\mathcal{D}}\; g & = & \mu \,;\, Ff \,;\, \gamma \,;\, Gg \,;\, \nu & \text{(explained below)} \\
id_{\mathcal{D}} & = & \eta\,, &
\end{array}$$

where natural transformations γ, η, μ, ν are assumed to exist:

$$\begin{array}{lcll}
\gamma & : & FG \mathrel{\dot\to} GF & F,G\text{-commuting transformation} \\
\eta & : & F \mathrel{\dot\to} G & F\text{- to } G\text{-unit transformation} \\
\mu & : & F \mathrel{\dot\to} FF & F\text{-generating transformation} \\
\nu & : & GG \mathrel{\dot\to} G & G\text{-reducing transformation} \quad .
\end{array}$$

Notice that G and ν play the role of F and μ in the usual nomenclature for monads and the Kleisli construction. The above definition of $f \;;_{\mathcal{D}}\; g$ is the simplest general way to combine f and g in that order into an arrow of type $Fa \to Gc$. Indeed, in order that a transformed f and transformed g both have the same ingredients in their target and source type, respectively, F has to be applied to f and G to g; this gives $Ff\colon FFa \to FGb$ and $Gg\colon GFb \to GGc$:

$$Fa \xrightarrow{?} FFa \xrightarrow{Ff} FGb \xrightarrow{?} GFb \xrightarrow{Gg} GGb \xrightarrow{?} Gb \quad .$$

Now an arrow of type $Fa \to Gc$ may be obtained by an F-generating transformation first, followed by Ff and Gg with an F,G-commuting transformation in between, and a G-reducing transformation at the end:

$$Fa \xrightarrow{\mu} FFa \xrightarrow{Ff} FGb \xrightarrow{\gamma} GFb \xrightarrow{Gg} GGb \xrightarrow{\nu} Gb \quad .$$

Clearly, the typing axioms, except for uniqueness of typing, are satisfied:

$$\begin{array}{l}
f\colon a \to_{\mathcal{D}} b, \quad g\colon b \to_{\mathcal{D}} c \quad \Rightarrow \quad f \;;_{\mathcal{D}}\; g\colon a \to_{\mathcal{D}} c \\
id_{\mathcal{D},a}\colon a \to_{\mathcal{D}} a \quad .
\end{array}$$

So in order to prove that $\mathcal{D}$ is a pre-category, it remains to show that $;_{\mathcal{D}}$ is associative and $id_{\mathcal{D}}$ is neutral for $;_{\mathcal{D}}$. We shall now give those proofs, assuming suitable properties on

γ, η, μ, ν along the way. First, for associativity, let $f\colon a \to_{\mathcal{D}} b$, $g\colon b \to_{\mathcal{D}} c$, and $h\colon c \to_{\mathcal{D}} d$ be arbitrary; then:

$$f \,;_{\mathcal{D}}\, (g \,;_{\mathcal{D}}\, h) = (f \,;_{\mathcal{D}}\, g) \,;_{\mathcal{D}}\, h$$

$\equiv$ definition $\mathcal{D}$

$$\begin{array}{ll} Fa \xrightarrow{\mu} FFa \xrightarrow{Ff} FGb \xrightarrow{\gamma} GFb \xrightarrow{G\mu} GFFb \xrightarrow{GFg} GFGc \xrightarrow{G\gamma} GGFc \xrightarrow{GGh} GGGd \xrightarrow{G\nu} GGd \xrightarrow{\nu} Gd & \\ \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| & \text{commutes} \\ Fa \xrightarrow{\mu} FFa \xrightarrow{F\mu} FFFa \xrightarrow{FFf} FFGb \xrightarrow{F\gamma} FGFb \xrightarrow{FGg} FGGc \xrightarrow{F\nu} FGc \xrightarrow{\gamma} GFc \xrightarrow{Gh} GGd \xrightarrow{\nu} Gd & \end{array}$$

(∗) $\Leftarrow$ { in order to split the terms into several parts at isomorphic objects: }
assume γ has inverse $\breve{\gamma}$, so that $\gamma F\colon FGF \dot{\rightarrow} GFF$ has inverse $\breve{\gamma}F$

$$\begin{array}{ll} Fa \xrightarrow{\mu} FFa \xrightarrow{Ff} FGb \xrightarrow{\gamma} GFb \xrightarrow{G\mu} GFFb \xrightarrow{GFg} GFGc \xrightarrow{G\gamma} GGFc \xrightarrow{GGh} GGGd \xrightarrow{G\nu} GGd \xrightarrow{\nu} Gd & \\ \| \qquad\qquad\qquad\qquad\qquad\qquad \gamma F \| \breve{\gamma}F \quad \breve{\gamma}G \| \gamma G \qquad\qquad\qquad\qquad\qquad\qquad \| & \text{commutes} \\ Fa \xrightarrow{\mu} FFa \xrightarrow{F\mu} FFFa \xrightarrow{FFf} FFGb \xrightarrow{F\gamma} FGFb \xrightarrow{FGg} FGGc \xrightarrow{F\nu} FGc \xrightarrow{\gamma} GFc \xrightarrow{Gh} GGd \xrightarrow{\nu} Gd & \end{array}$$

$\equiv$ diagram notation

$$\mu \,;\, Ff \,;\, \gamma \,;\, G\mu = \mu \,;\, F\mu \,;\, FFf \,;\, F\gamma \,;\, \gamma F \quad \wedge$$
$$GFg \,;\, \breve{\gamma}G = \breve{\gamma}F \,;\, FGg \quad \wedge$$
$$\gamma G \,;\, G\gamma \,;\, GGh \,;\, G\nu \,;\, \nu = F\nu \,;\, \gamma \,;\, Gh \,;\, \nu$$

$\equiv$ { for readability: }
define $\gamma_{2,1} = F\gamma \,;\, \gamma F\colon FFG \dot{\rightarrow} GFF$ and
define $\gamma_{1,2} = \gamma G \,;\, G\gamma\colon FGG \dot{\rightarrow} GGF$

$$\mu \,;\, Ff \,;\, \gamma \,;\, G\mu = \mu \,;\, F\mu \,;\, FFf \,;\, \gamma_{2,1} \quad \wedge$$
$$GFg \,;\, \breve{\gamma}G = \breve{\gamma}F \,;\, FGg \quad \wedge$$
$$\gamma_{1,2} \,;\, GGh \,;\, G\nu \,;\, \nu = F\nu \,;\, \gamma \,;\, Gh \,;\, \nu$$

$\equiv$ 2nd conjunct: naturality $\breve{\gamma}\colon GF \dot{\rightarrow} FG$;
1st, 3rd conjunct: { aiming at the next two steps, }
assume $\gamma \,;\, G\mu = \mu G \,;\, \gamma_{2,1}$ **and** $F\nu \,;\, \gamma = \gamma_{1,2} \,;\, \nu F$

$$\mu \,;\, Ff \,;\, \mu G \,;\, \gamma_{2,1} = \mu \,;\, F\mu \,;\, FFf \,;\, \gamma_{2,1} \quad \wedge$$
$$\gamma_{1,2} \,;\, GGh \,;\, G\nu \,;\, \nu = \gamma_{1,2} \,;\, \nu F \,;\, Gh \,;\, \nu$$

$\equiv$ naturality $\mu\colon F \dot{\rightarrow} FF$ and $\nu\colon GG \dot{\rightarrow} G$

$$\mu \,;\, \mu F \,;\, FFf \,;\, \gamma_{2,1} = \mu \,;\, F\mu \,;\, FFf \,;\, \gamma_{2,1} \quad \wedge$$
$$\gamma_{1,2} \,;\, GGh \,;\, G\nu \,;\, \nu = \gamma_{1,2} \,;\, GGh \,;\, \nu G \,;\, \nu$$

$\Leftarrow$ Leibniz

$$\mu \,;\, \mu F = \mu \,;\, F\mu \quad \wedge$$
$$G\nu \,;\, \nu = \nu G \,;\, \nu$$

$\equiv$ **assume** $\mu \,;\, \mu F = \mu \,;\, F\mu$ **and** $G\nu \,;\, \nu = \nu G \,;\, \nu$

true .

The $\gamma_{2,1}$ and $\gamma_{1,2}$ defined above are instances of a more general $\gamma_{m,n}$: $F^m G^n \dot{\to} G^n F^m$ that one may define easily by induction on m and n in several distinct but semantically equal ways. In step $(*)$ there is no other nontrivial way to split the giant terms into several parts than the way indicated: in general two intermediate objects can be isomorphic only if their denotations contain the same ingredients. More precisely, in step $(*)$ we have assumed that $FGFb$ and $GFFb$ are isomorphic via γF; an alternative is to assume that $FFGb$ and $GFFb$ are isomorphic, via $\gamma_{2,1}$. But since $\gamma_{2,1} = F\gamma \mathbin{;} \gamma F$ and since the term $F\gamma$ is present at just the right place, this alternative assumption is equivalent to ours.

Here are the five assumptions made along the way (the equalities are assumed, the typing is provable):

$$
\begin{array}{llllll}
\gamma \text{ has inverse } \breve{\gamma} & & & & \\
\gamma \mathbin{;} G\mu & = & \mu G \mathbin{;} \gamma_{2,1} & : & FG \dot{\to} GFF \\
F\nu \mathbin{;} \gamma & = & \gamma_{1,2} \mathbin{;} \nu F & : & FGG \dot{\to} GF
\end{array}
$$

and

$$
\begin{array}{lllll}
\mu \mathbin{;} \mu F & = & \mu \mathbin{;} F\mu & : & F \dot{\to} FF \\
G\nu \mathbin{;} \nu & = & \nu G \mathbin{;} \nu & : & GG \dot{\to} G
\end{array}
$$

where

$$
\begin{array}{lllll}
\gamma_{2,1} & = & F\gamma \mathbin{;} \gamma F & : & FFG \dot{\to} GFF \\
\gamma_{1,2} & = & \gamma G \mathbin{;} G\gamma & : & FGG \dot{\to} GGF \quad .
\end{array}
$$

Taking $F = \mathit{Id}$ and also $\mu, \gamma = \mathit{id}, \mathit{id}$, these assumptions specialise to those that make the Kleisli composition for G, ν associative.

The derivation of *nice* assumptions on η in order that $\mathit{id}_{\mathcal{D}}$ is neutral for $;_{\mathcal{D}}$ is problematic. Let us first consider only the first equality in "$f \mathbin{;_{\mathcal{D}}} \mathit{id} = \mathit{id} \mathbin{;_{\mathcal{D}}} f = f$". So, let f: $a \to_{\mathcal{D}} b$ be arbitrary; then:

$$
\begin{array}{lll}
 & & f \mathbin{;_{\mathcal{D}}} \mathit{id}_{\mathcal{D}} = \mathit{id}_{\mathcal{D}} \mathbin{;_{\mathcal{D}}} f \\
 & \equiv & \quad \text{definition } \mathcal{D} \\
 & & Fa \xrightarrow{\mu} FFa \xrightarrow{Ff} FGb \xrightarrow{\gamma} GFb \xrightarrow{G\eta} GGb \xrightarrow{\nu} Gb \quad = \\
 & & Fa \xrightarrow{\mu} FFa \xrightarrow{F\eta} FGa \xrightarrow{\gamma} GFa \xrightarrow{Gf} GGb \xrightarrow{\nu} Gb \\
 & \Leftarrow & \quad \text{Leibniz } \{\text{no nontrivial intermediate isomorphism seems plausible}\} \\
(*) & & Ff \mathbin{;} \gamma \mathbin{;} G\eta = F\eta \mathbin{;} \gamma \mathbin{;} Gf \\
 & \Leftarrow & \quad \text{naturality } \eta\text{: } F \dot{\to} G\text{, so } Ff \mathbin{;} \eta G = \eta F \mathbin{;} Gf \\
(\star) & & \gamma \mathbin{;} G\eta = \eta G \quad \wedge \quad F\eta \mathbin{;} \gamma = \eta F \quad .
\end{array}
$$

Equation $(*)$, for all f, is acceptably nice: it asserts a sort of naturality $F \dot{\to} G$. Since we have already assumed natural transformation η: $F \dot{\to} G$, it seems reasonable to require that $\gamma \mathbin{;} G\eta$ *is* ηG, as in line $(\star)$. However, when instantiating with $F, \gamma := \mathit{Id}, \mathit{id}$, both line $(*)$ and line $(\star)$ give requirements that are stronger than those for a monad: line $(*)$ becomes $f \mathbin{;} G\eta = \eta \mathbin{;} Gf$ (which is *not* just naturality η: $\mathit{Id} \dot{\to} G$), and line $(\star)$ becomes

$G\eta = \eta G$. So, we look for another sufficient condition that implies $f \mathbin{;} id_{\mathcal{D}} = id_{\mathcal{D}} \mathbin{;} f$ for all $f\colon\ a \to_{\mathcal{D}} b$. The following line of reasoning has been suggested by Lambert Meertens.

$$
\begin{array}{ll}
 & f \mathbin{;_{\mathcal{D}}} id_{\mathcal{D}} = id_{\mathcal{D}} \mathbin{;_{\mathcal{D}}} f \\
\equiv & \qquad \text{definition } \mathcal{D} \\
 & Fa \xrightarrow{\mu} FFa \xrightarrow{Ff} FGb \xrightarrow{\gamma} GFb \xrightarrow{G\eta} GGb \xrightarrow{\nu} Gb \quad = \\
 & Fa \xrightarrow{\mu} FFa \xrightarrow{F\eta} FGa \xrightarrow{\gamma} GFa \xrightarrow{Gf} GGb \xrightarrow{\nu} Gb \\
\equiv & \qquad \{\text{in order to shift } Ff \text{ and } Gf \text{ towards each other,}\} \\
 & \qquad \textbf{assume } \gamma \mathbin{;} G\eta \mathbin{;} \nu = \eta G \mathbin{;} \nu \textbf{ and } \mu \mathbin{;} F\eta \mathbin{;} \gamma = \mu \mathbin{;} \eta F \\
 & \mu \mathbin{;} Ff \mathbin{;} \eta G \mathbin{;} \nu = \mu \mathbin{;} \eta F \mathbin{;} Gf \mathbin{;} \nu \\
\Leftarrow & \qquad \text{Leibniz} \\
 & Ff \mathbin{;} \eta G = \eta F \mathbin{;} Gf \\
\equiv & \qquad \text{naturality } \eta\colon\ F \mathrel{\dot{\to}} G \\
 & \mathit{true} \quad .
\end{array}
$$

The two assumptions are acceptably nice.

As regards to the second equality in "$f \mathbin{;_{\mathcal{D}}} id = id \mathbin{;_{\mathcal{D}}} f = f$" we argue as follows. Let $f\colon\ a \to_{\mathcal{D}} b$ be arbitrary, then:

$$
\begin{array}{lll}
 & & f \mathbin{;_{\mathcal{D}}} id_{\mathcal{D}} = id_{\mathcal{D}} \\
 & \equiv & \qquad \text{definition } \mathcal{D} \\
 & & \mu \mathbin{;} Ff \mathbin{;} \gamma \mathbin{;} G\eta \mathbin{;} \nu = f \\
 & \equiv & \qquad \text{assumption in previous calculation} \\
(*) & & \mu \mathbin{;} Ff \mathbin{;} \eta G \mathbin{;} \nu = f \\
 & \equiv & \qquad \text{naturality } \eta\colon\ F \mathrel{\dot{\to}} G \\
(\star) & & \mu \mathbin{;} \eta F \mathbin{;} Gf \mathbin{;} \nu = f \quad .
\end{array}
$$

Thus we are led to assume the equation of line $(*)$ (for all $f\colon\ a \to_{\mathcal{D}} b$), or equivalently, of line $(\star)$. However, both equations are too complicated to be called nice; in particular, the "f" occurs in the *middle* of the term and not at one end, as in naturality assertions. Fortunately, the instantiation with $F, \gamma, \mu := Id, id, id$ does give a monad law:

$$
\begin{array}{ll}
 & \mu \mathbin{;} Ff \mathbin{;} \eta G \mathbin{;} \nu = f \qquad \text{for all } f\colon\ a \to_{\mathcal{D}} b \\
\equiv & \qquad \text{substitution } F, \gamma, \mu := Id, id, id \\
 & f \mathbin{;} \eta G \mathbin{;} \nu = f \qquad \text{for all } f\colon\ a \to Gb \\
\equiv & \qquad \text{for } \Leftarrow\text{: Leibniz;} \\
 & \qquad \text{for } \Rightarrow\text{: take } a, f := Gb, id_{Gb} \\
 & \eta G \mathbin{;} \nu = idG \quad .
\end{array}
$$

Summary. Let F, G be functors. Then we call $(F, G, \gamma, \eta, \mu, \nu)$ a **dyad** if:

$$\begin{array}{lll}
\gamma & : & FG \mathrel{\dot{\rightarrow}} GF \\
\eta & : & F \mathrel{\dot{\rightarrow}} G \\
\mu & : & F \mathrel{\dot{\rightarrow}} FF \\
\nu & : & GG \mathrel{\dot{\rightarrow}} G \quad ,
\end{array}$$

satisfy the following conditions:

$$\begin{array}{lllll}
\mu \mathbin{;} \mu F & = & \mu \mathbin{;} F\mu & : & F \mathrel{\dot{\rightarrow}} FF \\
G\nu \mathbin{;} \nu & = & \nu G \mathbin{;} \nu & : & GG \mathrel{\dot{\rightarrow}} G \\
\gamma \mathbin{;} G\eta \mathbin{;} \nu & = & \eta G \mathbin{;} \nu & & \\
\mu \mathbin{;} F\eta \mathbin{;} \gamma & = & \mu \mathbin{;} \eta F & & \\
\mu \mathbin{;} Ff \mathbin{;} \eta G \mathbin{;} \nu & = & f & \multicolumn{2}{l}{\text{for all } f\colon\ a \rightarrow_{\mathcal{D}} b} \\
\multicolumn{5}{l}{\gamma \text{ has inverse } \breve{\gamma}} \\
\gamma \mathbin{;} G\mu & = & \mu G \mathbin{;} \gamma_{2,1} & : & FG \mathrel{\dot{\rightarrow}} GFF \\
F\nu \mathbin{;} \gamma & = & \gamma_{1,2} \mathbin{;} \nu F & : & FGG \mathrel{\dot{\rightarrow}} GF
\end{array}$$

where

$$\begin{array}{lllll}
\gamma_{2,1} & = & F\gamma \mathbin{;} \gamma F & : & FFG \mathrel{\dot{\rightarrow}} GFF \\
\gamma_{1,2} & = & \gamma G \mathbin{;} G\gamma & : & FGG \mathrel{\dot{\rightarrow}} GGF \quad .
\end{array}$$

After substituting $F, \gamma, \mu := \mathit{Id}, \mathit{id}, \mathit{id}$, these requirements are equivalent to the statement that (G, η, ν) is a monad.

References

[1] M. Barr and C. Wells. *Category Theory for Computing Science*. Prentice Hall, 1990.

[2] P. Wadler. Comprehending monads. In *ACM Conference on Lisp and Functional Programming*, June 1990. To appear in Mathematical Structures in Computer Science.