Adjunctions formulated as cata/anamorphisms

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Abstract

The following is nothing but a formulation of the notion of *adjunction* in a terminology that is close to our current conventions in Constructive Algorithmics. It has been proposed by Lambert Meertens.

Theorem The five statements below are equivalent; each of them is a definition of "F is adjoint to G".

Remark. Statement 1 does not give immediately the full consequences of F being adjoint to G (and, in fact, none of the statements does), but the formula is so simple that it is easy to remember and to reconstruct from it the other statements, in particular Statement 2. Statements 2 and 3 are the ones that are so appealing for The Constructive Algorithmicians in view of the similarity to the notions and properties of catamorphisms and anamorphisms. The other statements have no obvious analogues in terms of cata/anamorphisms.

1. $(\mathsf{F}{\circ\!\!\rightarrow}) \cong ({\circ\!\!\rightarrow}\mathsf{G})$.

   Explanation. $\circ\!\!\rightarrow$ is the envelope functor of type $\mathbf{K}^{op} \times \mathbf{K} \to \text{Set}$ defined by

   $$\begin{array}{lcl} A \circ\!\!\rightarrow B & = & \text{the set of all morphisms } A \to B \text{ in } \mathbf{K}\text{, i.e., } Hom_{\mathbf{K}}(A,B) \\ f \circ\!\!\rightarrow g & = & (\lambda\phi :: f; \phi; g) \quad : \quad (B \circ\!\!\rightarrow C) \to (A \circ\!\!\rightarrow D) \end{array}$$

   for $f : A \to B$ in $\mathbf{K}$ and $g : C \to D$ in $\mathbf{K}$. Further, $(\mathsf{F}{\circ\!\!\rightarrow})$ is an ad-hoc notation for the (infix written) bi-functor $x(\mathsf{F}{\circ\!\!\rightarrow})y = x\mathsf{F} \circ\!\!\rightarrow y$, and similarly $x({\circ\!\!\rightarrow}\mathsf{G})y = x \circ\!\!\rightarrow y\mathsf{G}$. So, if $\mathsf{F} : \mathbf{C} \to \mathbf{D}$ and $\mathsf{G} : \mathbf{D} \to \mathbf{C}$, then both $(\mathsf{F}{\circ\!\!\rightarrow})$ and $({\circ\!\!\rightarrow}\mathsf{G})$ go from $\mathbf{C}^{op} \times \mathbf{D}$ to Set .

   Since both operands of $\cong$ are functors, the isomorphism between the two functors apparently takes place in the category $[\mathbf{C}^{op} \times \mathbf{D} \dot{\to} \text{Set}]$ of functors, where morphisms are natural transformations. Thus the isomorphism asserts the existence of natural transformations; these are denoted $(\!|\_|\!)$ and $[\![\_]\!]$ in the following formulations. Notice however that these are not exactly the same as the conventional catamorphisms and anamorphisms!

2. There exist $(\!|\_|\!)$ and $[\![\_]\!]$ such that

   (a) $(\!| f; \phi; g\mathsf{G} |\!) \;=\; f\mathsf{F}; (\!|\phi|\!); g$

   (b) $[\![f\mathsf{F}; \psi; g]\!] \;=\; f; [\![\psi]\!]; g\mathsf{G}$

   (c) $(\!|\_|\!)$ and $[\![\_]\!]$ are each others inverse.

Remark. This statement can be read as a "Promotion (or better: Fusion) Property" of $(\!|\_|\!)$ and $[\![\_]\!]$. As shown below it is just Statement 1 worked out in detail. Actually, given (c), assertions (a) and (b) are equivalent so that one of them suffices.

Warning. We have suppressed the subscript X, Y to the natural transformations $(\!|\_|\!)$ and $[\![\_]\!]$. When $\mathbf{C} = \mathbf{D}$ this is no problem since then they can be recovered by the usual type-inference. However, the subscripts are really needed when $\mathbf{C} \neq \mathbf{D}$; this occurs in the example below. This warning also applies to all following statements.

3. There exist $(\!|\_|\!)$, $[\![\_]\!]$, ε and η such that

   (d) $\quad h = (\!|\phi|\!) \quad \equiv \quad \eta ; h\mathsf{G} = \phi$

   (e) $\quad h = [\![\psi]\!] \quad \equiv \quad h\mathsf{F} ; \varepsilon = \psi$

   (f) $\quad (\!|\_|\!)$ and $[\![\_]\!]$ are each others inverse.

   Remark. This statement can be read as a "Uniqueness Property" of $(\!|\_|\!)$ and $[\![\_]\!]$. Here η and ε play the role of *in* and *out* respectively. For example, compare it to the characterisation of real catamorphisms, here denoted $\text{cata}(\_)$:

   $(*) \quad h = \text{cata}(\phi) \quad \equiv \quad in ; h\mathsf{I} = h\mathsf{F} ; \phi$

   Taking $\mathsf{G} := \mathsf{I}$ in (d) and $\mathsf{F} := A$ in $(*)$ makes both equivalences identical.

   As a corollary we obtain by substituting $h, \psi := \phi, (\!|\phi|\!)$ in (e), and similarly in (d), that

   $$(\!|\phi|\!) = \phi\mathsf{F} ; \varepsilon \qquad \text{hence} \qquad \varepsilon = (\!|\mathsf{id}|\!)$$
   $$[\![\psi]\!] = \eta ; \psi\mathsf{G} \qquad \text{hence} \qquad \eta = [\![\mathsf{id}]\!] .$$

   Again, given (f), assertions (d) and (e) are equivalent so that one of them suffices.

4. There exist ε and η such that

   (g) $\quad \eta ; \phi\mathsf{G} = \psi \quad \equiv \quad \psi\mathsf{F} ; \varepsilon = \phi .$

   Remark. This formulation, and the next one, are conventional category theoretic formulations of the notion of adjunction.

5. There exist ε and η such that

   (h) $\quad \varepsilon \quad : \quad \mathsf{GF} \stackrel{\cdot}{\to} \mathsf{I}$

   (i) $\quad \eta \quad : \quad \mathsf{I} \stackrel{\cdot}{\to} \mathsf{FG}$

   (j) $\quad \mathsf{id} \quad = \quad \eta ; \varepsilon\mathsf{G}$

   (k) $\quad \mathsf{id} \quad = \quad \eta\mathsf{F} ; \varepsilon .$

Typechecking these statements reveals that in each of them

$\mathsf{F}, (\!\|\_\|\!) : \mathbf{C} \to \mathbf{D}$	in fact	$(\!\|\_\|\!)_{X,Y} : (X \to Y\mathsf{G}) \to (X\mathsf{F} \to Y)$ for all $X \in \mathbf{C}$ and $Y \in \mathbf{D}$	
$\mathsf{G}, [\![\_]\!] : \mathbf{D} \to \mathbf{C}$	in fact	$[\![\_]\!]_{X,Y} : (X\mathsf{F} \to Y) \to (X \to Y\mathsf{G})$ for all $X \in \mathbf{C}$ and $Y \in \mathbf{D}$	
$f, \eta, \phi \in \mathbf{C}$	in fact	$\phi : \text{Target}\, f \to \text{Source}\, g\mathsf{G}$	$\eta_X : X \to X\mathsf{FG}$
$g, \varepsilon, \psi \in \mathbf{D}$	in fact	$\psi : \text{Target}\, f\mathsf{F} \to \text{Source}\, g$	$\varepsilon_Y : Y\mathsf{GF} \to Y .$

The proofs of the equivalences are straightforward calculations; they can be given as a loop of implications. (Even the implication from 5 to 1 is simple and elegant!) We show only the implication from Statement 1 to Statement 2:

$(\mathsf{F}{\circ}{\rightarrow}) \cong ({\circ}{\rightarrow}\mathsf{G})$

$\equiv$ unfold, calling the ntrfs $[\![\_]\!]$ resp $(\!|\_|\!)$

$[\![\_]\!] : (\mathsf{F}{\circ}{\rightarrow}) \dot{\rightarrow} ({\circ}{\rightarrow}\mathsf{G})$

$(\!|\_|\!) : ({\circ}{\rightarrow}\mathsf{G}) \dot{\rightarrow} (\mathsf{F}{\circ}{\rightarrow})$

$(\!|\_|\!)$ and $[\![_]\!]$are each others inverse

$\Rightarrow$ concentrating on $(\!|\_|\!)$ only; for all f, g

$f({\circ}{\rightarrow}\mathsf{G})g ; (\!|\_|\!) = (\!|\_|\!) ; f(\mathsf{F}{\circ}{\rightarrow})g$

$\equiv$ extensionality: for all ϕ in the appropriate set

$\phi.\ (f({\circ}{\rightarrow}\mathsf{G})g ; (\!|\_|\!)) = \phi.\ ((\!|\_|\!) ; f(\mathsf{F}{\circ}{\rightarrow})g)$

$\equiv$ unfold

$(\!|f ; \phi ; g\mathsf{G}|\!) = f\mathsf{F} ; (\!|\phi|\!) ; g$

and similarly for $[\![\_]\!]$.

* * *

The adjunction of F to G is sometimes denoted schematically by

$$\frac{X \rightarrow Y\mathsf{G}}{X\mathsf{F} \rightarrow Y}$$

This is formula $(\mathsf{F}{\circ}{\rightarrow}) \cong ({\circ}{\rightarrow}\mathsf{G})$ slightly (and uncompletely) worked out, and says that for any X , Y , and $\phi : X \rightarrow Y\mathsf{G}$ there exists a special morphism $X\mathsf{F} \rightarrow Y$, namely $(\!|\phi|\!)_{X,Y}$, and conversely for any X , Y , and $\psi : X\mathsf{F} \rightarrow Y$ there exists a special morphism $X \rightarrow Y\mathsf{G}$ namely $[\![\psi]\!]_{X,Y}$. The "speciality" is understood to be the properties of Statement 2. Clearly, Statement 1 is much more concise and precise; the advantage (or rather, disadvantage) of the schematic indication is that F and G can be left unnamed: the X and $X\mathsf{F}$ (just an expression in X) together suggest a definition for F, and the Y and $Y\mathsf{G}$ (just an expression in Y) together suggest a definition for G . Notice that the top line is entirely in $\mathbf{C}$, and the bottom line in $\mathbf{D}$.

Example: Unique Extension and Free Algebras The unique extension property (*in the sense as used in algebra and category theory*) can be phrased as an adjunction. To do so we first explain a bit of notation.

Let $\mathbf{K}$ be a category and H be an endo-functor on $\mathbf{K}$. An H-algebra is a morphism $\phi : A\mathsf{H} \rightarrow A$ where A is the target of ϕ . An H-homomorphism h from H-algebra ϕ to H-algebra ψ (denoted $h : \phi \overset{\mathsf{H}}{\rightarrow} \psi$), is a morphism $h : \mathrm{Target}\,\phi \rightarrow \mathrm{Target}\,\psi$ with the additional property that $\phi ; h = h\mathsf{H} ; \psi$. For a functor H we let $\mu\mathsf{H}$ denote the initial H-algebra. (We assume that $\mathbf{K}$ and H are such that $\mu\mathsf{H}$ exists.) For any object A we let A also denote the constant functor with $xA = A$ for any object x and $xA = \mathsf{id}_A$ for any morphism x . Further we define monofunctor $A{+}\mathsf{H}$ by $x(A{+}\mathsf{H}) = xA + x\mathsf{H}$ for all objects and morphisms x . Notice that any morphism $\phi : X(A + \mathsf{H}) \rightarrow X$ can be written as $\phi = \psi \triangledown \chi$ with $\psi : A \rightarrow X$ and $\chi : X\mathsf{H} \rightarrow X$. This holds also for $\mu(A + \mathsf{H})$, and we define $\mathsf{tau}_A, \mathsf{join}_A$

by $\mathsf{tau}_A \triangledown \mathsf{join}_A = \mu(A + \mathsf{H})$. In view of the particular form of functor $A + \mathsf{H}$ (called *free* and formerly *factorable* by Malcolm) we can define maps and reduces as usual, denoted by $*$ and $/$ (thus suppressing the dependency on H).

Now we claim that the following adjunction exists.

$$\frac{A \to \mathrm{Target}\,\phi}{\mathsf{join}_A \overset{\mathsf{H}}{\to} \phi}$$

The category of the top line is $\mathbf{K}$ (e.g., Set); the category of the bottom line is "H-algebras over $\mathbf{K}$". Functors F and G of this adjunction have been left anonymous. Apparently functor F is given by $A\mathsf{F} = \mathsf{join}_A$ (which is an H-algebra indeed) for any object A; for functions $f : A \to B$ we take $f\mathsf{F} = f* : \mathsf{join}_A \overset{\mathsf{H}}{\to} \mathsf{join}_B$, the usual f-map. (In the absence of any knowledge about H I see no other way to define $f\mathsf{F}$.) Similarly, for any H-algebra ϕ we have, apparently, that $\phi\mathsf{G} = \mathrm{Target}\,\phi$; for homomorphisms $h : \phi \overset{\mathsf{H}}{\to} \psi$ we take $h\mathsf{G} =$ the morphism $h : \mathrm{Target}\,\phi \to \mathrm{Target}\,\psi$. Thus G is the forgetful functor (it forgets about the commutative property $\phi; h = h\mathsf{H}; \psi$), usually denoted U.

The assertion, then, is that for each A, ϕ and $f : A \to \mathrm{Target}\,\phi$ there exists a "special" $(\!|f|\!) : \mathsf{join}_A \overset{\mathsf{H}}{\to} \phi$, and for each A, ϕ, and $h : \mathsf{join}_A \overset{\mathsf{H}}{\to} \phi$ there exists a "special" $[\![h]\!] : A \to \mathrm{Target}\,\phi$. Indeed, we can take $(\!|f|\!) =$ "the real catamorphism $f * ; \phi/$", and we can take $[\![h]\!] =$ "$\mathsf{tau}_A; h$ considered as a morphism $: A \to \mathrm{Target}\,\phi$". It is then not hard to verify the properties of Statement 2. It turns out that that $\eta_A = \mathsf{tau}_A$ and $\varepsilon_\phi = \phi/$. Notice that the subscript to ε plays an essential role!

Thus we have *proved* the adjunction. Conversely, if the forgetful functor G is *postulated* to have a left adjoint, F say, then this postulation asserts the existence of ε_ϕ (which we might call the *reduce* with ϕ and denote it by $\phi/$), the existence of $f\mathsf{F}$ (which we might call *f-map* and denote it by $f*$), the existence of $A\mathsf{F}$ (which we might call *the free H-algebra over A* and denote it by join_A), and the existence of η_A (which we might call *injection* or singleton structure former, and denote it by tau_A) that satisfy the usual laws for maps and reduces (implying amongst others that $(*, \mathsf{tau}, \mathsf{join}/)$ is a monad).

Since for given A and ϕ the $(\!|f|\!) : \mathsf{join}_A \overset{\mathsf{H}}{\to} \phi$ is uniquely determined by $f : A \to \mathrm{Target}\,\phi$ and extends f in the sense that (see d)

$$\mathsf{tau}_A; (\!|f|\!) \quad = \quad f\,,$$

the adjunction is known as "the unique extension" property.